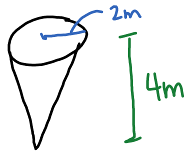
# Lecture 9: Applications of Differentiation

October 12, 2016 2:32 PM

**Related rates**: an application of the chain rule

- relate a desired quantity to one that's easy to measure

Ex: water tank



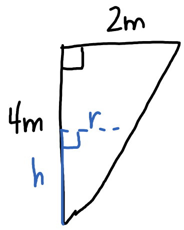

we pump water into the tank at a rate of $2m^3$/min

Find the rate at which the water is rising at 3m deep.
(rate of change of the water level becomes less the higher the water surface is; ie. cone gets wider at the top)

| Information that we have: | Information that we want: |
|---|---|
| • change of volume | • rate of change of height |

$V = \frac{1}{3}\pi \cdot r^2 \cdot h$ <- function relating volume and height

rate of change: **derivatives**




$$\frac{2m}{4m} = \frac{r}{h}$$
$$r = \frac{h}{2}$$

New formula: $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h = \frac{1}{12}\pi h^3$

Take derivative:
(t: time)

$$\frac{dV}{dt} = \frac{\pi}{12} * 3h^2 * \frac{dh}{dt}$$

h is a function of time, height changes with time

We want: rate of change of height, express $\frac{dh}{dt}$explicitly:

$$\frac{dh}{dt} = \frac{12}{3\pi h^2} * \frac{dV}{dt} = \frac{4}{\pi h^2} * \frac{dV}{dt}$$

^ formula for rate of change of the height, depending on the current height and water flow ($\frac{dV}{dt}$)

Put h=3m and $\frac{dV}{dt} = 2m^3/min$ in formula:

$$\frac{dh}{dt} = \frac{4}{\pi(3\cancel{m})^2} * \frac{2m^{\cancel{3}}}{min}$$
$$= \frac{4}{\pi * 9} * 2\frac{m}{min} = \frac{8}{9\pi} m/min$$

**Linear approximations**

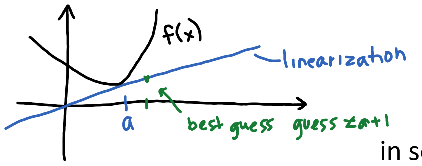
- application of finding tangent line equations

main idea: we use a linear function to approximate a very complicated function

$$\boxed{L_f(x) = f(x) \approx f(a) + f'(a)(x-a)}$$

**linear approximation** or **linearization** at a

idea:




in some applications, this guess is more than enough

Ex:

$f(x) = \sqrt{x+3}$, linearization at a=1

approximate: (1) sqrt(3.98)

(2) sqrt(4.05)

this is more calculator intensive than the exam

things to compute: f(1), f'(1)

$f(1) = \sqrt{1+3} = 2$

$f'(x) = \dfrac{1}{2\sqrt{x+3}}$

$f'(1) = \dfrac{1}{4}$

put formula together:

$L_f(x) = f(1) + f'(1)(x-1)$

$= 2 + \dfrac{1}{4}(x-1) = 2 + \dfrac{x}{4} - \dfrac{1}{4} = \boxed{\dfrac{7}{4} + \dfrac{x}{4}}$

Now approximate x=3.98:

$L_f(0.98)$

$f(x) = \sqrt{x+3}$, so to approximate $\sqrt{3.98}$, we need to set $x = 0.98$

not rounded

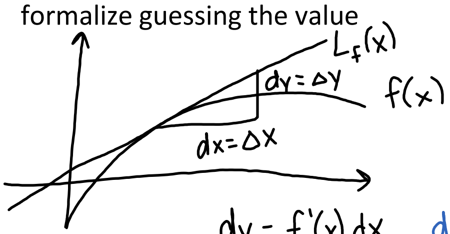
$L_f(0.98) = \dfrac{7}{4} + \dfrac{0.98}{4} = 1.995$ exact value: 1.994993734

approximate x=1.05 (for $\sqrt{4.05}$)

$L_f(1.05) = \dfrac{7}{4} + \dfrac{1.05}{4} = 2.0125$ exact value: 2.01246118

## Differentials

- formalize guessing the value




$dy = f'(x)\,dx$ differential

f'(x) is slope of f at x

Ex:

$f(x) = x^3 + x^2 - 2x + 1$

compare delta y and dy if x changes from 2 to 2.05

$f(2) = 2^{3}{}^{2}_{2} - 2 * 2 + 1 = 9$

$f(2.05) = 9.717625$

so:

$\Delta y = f(2.05) - f(2) = 0.717625$

so:
$\Delta y = f(2.05) - f(2) = 0.717625$

Now with differential:
$dy = f'(x)dx$
given:
$dx = 0.05$ (go from 2 to 2.05)
$f'(x) = 3x^2 + 2x - 2, \quad f'(2) = 14$

at x=2:
$dy = f'(2) * 0.05 = 14 * 0.05 = 0.7$

With this method, base point is flexible until the end. In the linearization, we need it at the beginning.

Same example, but approximate from 2 to 2.01
$dy = f'(2) * 0.01$ ← new dx
$= 14 * 0.01 = 0.14$ ← guess for value increase

$f(2.01) = 9.140701$ ← exact value of f
$f(2) = 9$, so exact value increase is $f(2.01) - f(2) = 0.140701$

Linearization is only a good approximation around x = a (base point) and not so good further away from x = a.

Ex:
The radius of a sphere is measured as 21cm with a possible error of 0.05cm.
max error in using this value to compute the volume

$V = \frac{4}{3}\pi r^3 \qquad dr = \Delta r$

$dV = \text{change in volume; depends on dr (change of measured radius)}$

r=21cm, dr=0.05
$dV = \frac{4}{3}\pi * 3r^2 * dr = 4\pi r^2 * dr = 4\pi(21)^2 * 0.05 \approx 277cm^3$
seems big, BUT, volume at radius 21:
$V = \frac{4}{3}\pi * 21^3, \qquad dV = 4\pi(21)^2 * 0.05$
$\frac{dV}{V} = \frac{\cancel{4\pi} * \cancel{21^2} * 0.05}{\cancel{\frac{4}{3}}\cancel{\pi}(21)^{\cancel{3}}} = \frac{\cancel{3}(0.05)}{\cancel{21}\ 7} = \frac{0.05}{7} = \frac{5}{700} = \frac{1}{140}$
$\approx 0.0072$
we make less than 1% error! (error is 0.72%)